A MATHEMATICAL CLUE TO THE SEPARATION PHENOMENA ON THE TWO-DIMENSIONAL NAVIER-STOKES EQUATION (Mathematical Analysis of Incompressible Flow)

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A MATHEMATICAL CLUE TO THE SEPARATION PHENOMENA ON THE TWO-DIMENSIONAL NAVIER-STOKES EQUATION

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1. INTRODUCTION

Ohya and Karasudani [8] developed a new wind turbine system that consists of a diffuser shroud with a broad-ring at the exit periphery and a wind turbine inside it. Their experiments show that a diffuser-shaped (not nozzle-shaped) structure can accelerate the wind at the entrance of the body. A strong vortex formation with a low-pressure region is created behind the broad brim. Accordingly, the wind flows into a low-pressure region, the wind velocity is accelerated further near the entrance of the diffuser. We would like to analyze this "wind-lens phenomena" in pure mathematical approach. For the first step, we need to figure out why the diffuser shroud creates vortices easier than the nozzle shroud. In general, creation of a vortex needs separation phenomena near a boundary (namely, topological changing phenomena), and before separating from the boundary, the flow moves toward reverse direction near the boundary against the laminar flow direction. There are several results related to the separation (in other words, wake region) in pure mathematics. Using the Oseen system is one of the mathematical approach to analyze the wake region. For the detailed discussion of the Oseen system, we refer the reader to [2]. In a convex obstacle case, the character of the system is elliptic in front of the obstacle. To the contrary, its character changes into parabolic type (wake region) behind the obstacle (see [4] for example). Maekawa [6] considered the two-dimensional Navier-Stokes equations in a half plane under the no-slip boundary condition. He established a solution formula for the vorticity equations and got a sufficient condition on the initial data for the vorticity to blow up to the inviscid limit. Ma and Wang [5] provided a characterization of the boundary layer separation of 2-D incompressible viscous fluids. They considered a separation equation linking a separation location and a time with the Reynolds number, the external forcing and the initial velocity field. However, none of the above studies has shown the mechanism behind the reverse flow phenomena (topological instability) rigorously. In this paper we show that a diffuser-shaped boundary induces the reverse flow even near the entrance of the diffuser. Let us be more precise. We consider the two-dimensional Navier-Stokes equation in $\Omega\subset\mathbb{R}^2$ (define Ω later) with no-slip and inflow-outflow conditions on $\partial\Omega$. We need to handle a shape of the boundary $\partial\Omega$ precisely, thus we set a parametrized smooth boundary $\varphi:[0,S]\to\mathbb{R}^2$ as $|\partial_s\varphi(s)|=1$, $|\partial_s^2\varphi(s)|=\kappa$ (curvature), $\varphi(0)=(0,0)$, $\partial_s\varphi(0)=(1,0)$, $\partial_s^2\varphi(0)=(0,-\kappa)$. We choose S later (should be sufficiently small). We define $n=n(s):=(\partial_s\varphi(s))^\perp$ as a unit normal vector and $\tau=\tau(s)=\partial_s\varphi(s)$ as a unit tangent vector, where $\perp$ represents upward direction.

In order to define the domain Ω, we need the following coordinate.

Definition 1.1. (Normal coordinate.) For $s \in [0, S]$ and $r \in [0, R]$, let

$$\Phi(s,r) = \Phi_\varphi(s,r) := n(s)r + \varphi(s).$$

Remark 1.2. Since $\partial_s n(s) = \kappa\tau(s)$ (Frenet-Serret formulas), we see that

$$(\partial_r\Phi)(s,r) = n(s) \quad \text{and} \quad (\partial_s\Phi)(s,r) = (r\kappa + 1)\tau(s).$$

Now we define the domain Ω as follows:

$$\Omega = \Omega_{S,R} := \{\Phi(s,r) \in \mathbb{R}^2 : s \in (0,S),\ r \in (0,R)\}.$$

Note that we take S and R to be sufficiently small depending on the initial data and the inflow condition. The non-stationary two-dimensional Navier-Stokes equation is expressed as

$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u = -\nabla p, \quad \operatorname{div} u = 0 \quad in \quad \Omega \subset \mathbb{R}^2, \\ u|_{\cup_{s=0}^{S}\varphi(s)} = 0, \end{cases} \tag{1.1}$$

where $u = u(x) = u(x,t) = (u^1(x_1,x_2,t), u^2(x_1,x_2,t))$. In this paper we sometimes abbreviate the time t not x. Let $\alpha_1, \alpha_2 > 0$ and $\alpha_3 \in \mathbb{R}$ be coefficients of the inflow condition (Poiseuille type flow profile which seems to be the most natural setting) such that

$$(u\cdot\tau)(\Phi(0,r)) = u^1(0,r,t) = \alpha_1 r - \frac{\alpha_2}{2}r^2 + \frac{\alpha_3}{3!}r^3 + O(r^4).$$

Also assume $(u\cdot n)(\Phi(0,r)) = u^2(0,r) = 0$. We assume that there exists a smooth solution except for the origin, namely, assume that there exists a pair of solution (u,p) to (1.1) in

$$u, p \in C^\infty([0,T]\times D) \cap C^\infty((0,T]\times(\Omega\setminus B_\epsilon)) \quad \text{for any} \quad D \Subset \Omega \quad \text{and} \quad \epsilon > 0,$$

where $B_\epsilon = \{x \in \mathbb{R}^2 : |x| < \epsilon\}$.

Remark 1.3. Combining a result of Navier-Stokes initial value problem in Lipschitz domain [7], and a boundary regularity result [3] (We believe we can generalize their result to various smooth domains), the above smoothness assumption should become true.

Remark 1.4. We can avoid interior blow-up by taking sufficiently small R. Thus we only need to care boundary regularity not interior regularity. Moreover we can also avoid boundary blow-up except for the origin by taking sufficiently small S. Thus it is reasonable to assume T to be sufficiently large (for sufficiently small S and R).

Definition 1.5. (Laminar flow.) u is "laminar flow" (near the origin) iff u is smooth (including the origin) in Ω, $|u(x)| \neq 0$ for $x \in \Omega$ and the flow u is to the rightward direction (laminar flow direction), namely,

$$(u\cdot\tau)(x) > 0$$

for $x \in \Omega$.

We mainly consider a geometrical shape of the laminar flow near the origin. In this case, one of the five situations only occur (for fixed time t): diffusing, almost parallel, concentrating laminar flows, topologically changing flow (inducing the reverse flow) or non-smoothness at the origin. Sometimes we write $u\cdot\tau = (u\cdot\tau)(s,r) = (u\cdot\tau)(s,r,t) = (u\cdot\tau)(x,t)|_{x=\Phi(s,r)}$ unless confusion occurs.

Definition 1.6. (Classification of Navier-Stokes flow for fixed time.) Let

$$\mathcal{L}_t(s,r) = \mathcal{L}(s,r) := (r\kappa + 1)\frac{u \cdot n}{u \cdot \tau} \quad \text{(slope of the velocity with Riemannian metric)}$$

and

$$\beta(t) := \lim_{s,r \to 0} \partial_s \partial_r \mathcal{L}(s,r).$$

- Diffusing laminar flow: We call diffusing laminar flow iff $u, p \in C^\infty(\Omega)$ and
$$\beta(t) > 0.$$
- Almost parallel laminar flow: We call almost parallel laminar flow iff $u, p \in C^\infty(\Omega)$ and
$$\beta(t) = 0.$$
- Concentrating laminar flow: We call concentrating laminar flow iff $u, p \in C^\infty(\Omega)$ and
$$\beta(t) < 0.$$
- Topologically changing flow (not laminar flow case): We say topologically changing flow iff $u, p \in C^\infty(\Omega)$ and there is $x \in \Omega$ such that $|u(x)| = 0$ or $(u \cdot \tau)(x) \leq 0$.
- Non-smoothness at the origin: We say non-smoothness at the origin iff
$$u(\cdot, t) \notin C^\infty(\Omega \cap B_\epsilon) \quad \text{or} \quad p(\cdot, t) \notin C^\infty(\Omega \cap B_\epsilon) \quad \text{for} \quad \epsilon > 0.$$

Remark 1.7. Rigorously, we need to quantify the first order diffusing/almost parallel/ concentrating laminar flows by using $\lim_{s,r\to 0} \partial_r \mathcal{L}(s,r)$, and we need to say the above definition $\beta(t)$ as the second order diffusing/almost parallel/ concentrating laminar flows. However, due to the inflow condition $(u \cdot n)(\Phi(0,r)) = 0$, namely, $\lim_{s\to 0} \mathcal{L} = 0$ for $r > 0$, the first order part is always zero: $\lim_{s,r\to 0} \partial_r \mathcal{L} = 0$. Thus in this paper, we do not distinguish "order" of the laminar flow (higher order part is not important in this paper).

Remark 1.8. If we consider the most general inflow condition, we need to handle $\lim_{s,r\to 0} \partial_r^2 \mathcal{L}(s,r)$ (quantified non-uniform structure). However we do not mention more in this paper. Note that non-uniform and the first order part are included in the inflow condition. But the second order part is not (it is included in the interior flow structure).

In order to give the main theorem, we need to define "trajectory".

Definition 1.9. (Trajectory.) Let $\tilde{\gamma}_X$ be in Ω and it satisfies

$$\partial_t \tilde{\gamma}_X(t) = u(\tilde{\gamma}_X(t), t), \quad \gamma_X(0) = X \in \Omega.$$

Note that the equation (1.1) can be rewritten as $\partial_t(u(\tilde{\gamma}(t), t)) = (\Delta u - \nabla p)(\tilde{\gamma}(t), t)$.

The following is the main theorem.

Theorem 1.10. *(Horizontally stopping particles phenomena.) Let the initial datum u_0 satisfies diffusing laminar flow condition, namely, $\beta(0) > 0$. If $\kappa\alpha_2 + \alpha_3 > 0$, then the topologically changing flow (or non-smoothness at the origin) must occur at finite time. In other words, particles near the boundary slow down and finally stop horizontally at finite time. More precisely, there is $\tilde{R} < R$ such that if $\tilde{r} < \tilde{R}$, then*

$$\lim_{t \to \tilde{T}} (u \cdot \tau)(\tilde{\gamma}_{\Phi(0,\tilde{r})}(t), t) = 0,$$

where $\tilde{T} < T$ is depending on $\tilde{r}$, ν, κ, α_1, α_2 and α_3.

Remark 1.11. A sort of Reynolds number does not appear in the above main theorem due to the inflow condition (Poiseuille type flow profile). If we relax the inflow condition, namely, we take non-uniform ($\lim_{s,r\to 0}\partial_r^2\mathcal{L}$) and first order ($\lim_{s,r\to 0}\partial_r\mathcal{L}$) parts into inflow condition to be nonzero, some kind of the Reynolds number should appear in the above main theorem.

Remark 1.12. There are direct and indirect evidences for the validity of the "Kutta condition" in restricted regions (see [1]). The method used in the above theorem may give another support for the validity of the Kutta condition in pure mathematical sense.

Now we give outline of the proof briefly. Basically, we need to estimate trajectory of a particle near the boundary. In order to do so, we need to estimate each Δu and ∇p near the boundary. First we construct " streamline coordinate " and then we estimate Δu directly. Next we construct " pressure coordinate " based on level set of the pressure and no-slip boundary condition. In this case, $\Delta u = \nabla p$ on the boundary is the crucial point. Third we calculate some kind of Riemannian metric of the " pressure coordinate " at the origin (the pressure is nonlocal operator, nevertheless we can estimate it by using orders of approximation).

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